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NOTE

THE SOLUTION OF A FAMOUS TWO-CENTURIES-OLD PROBLEM THE LEONHARD EULER-LATIN SQUARE CONJECTURE

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A little over 20 years ago three mathematicians, working in the United States, disproved a celebrated conjecture of the eminent 18th-century mathematician Leonhard Euler. The solution of this problem represents a classical example of how different areas of mathematics can be employed and integrated to arrive at a resolution of a long-standing conjecture, a conjecture which, in this case, arose out of the study of a type of magic square. The intricacies of magic squares attracted the attention of Leonhard Euler (1707-1783), who during the last years of his life wrote an extensive memoir on a new species of magic squares, now called Latin squares [Euler 1923].

A Latin square is an $n \times n$ matrix of n symbols, such that each symbol occurs once in every row and every column. The word Latin comes from Euler using Latin letters as the symbols for the first of his pair of orthogonal Latin squares. The particular Latin squares which Euler studied are orthogonal Latin squares.

Let A and B be two Latin squares of order n, and let a_{ij} and b_{ij}, $1 \leq i, j \leq n$, denote the entries in the ith row and the jth column in A and B, respectively. The two Latin squares A and B are said to be orthogonal if the n^2-ordered pairs (a_{ij}, b_{ij}), $1 \leq i, j \leq n$, are distinct. Thus, if the two squares are superimposed to form an $n \times n$ square with ordered pairs as entries, the orthogonality condition means that the entries of the resultant square are distinct. As an example, Fig. 1 shows two orthogonal Latin squares, and Fig. 2 shows the resultant square when they are superimposed. The resultant square is often called a Graeco-Latin square because of the terminology used by Euler, who used Greek letters for one square and Latin letters for the other square.

In Euler's day it was not difficult to prove that there are no orthogonal Latin squares of order 2 (2×2 squares). Although Graeco-Latin squares of orders 3, 4, and 5 were known, there re-

$$A = \begin{vmatrix} 312 \\ 231 \\ 123 \end{vmatrix} \qquad B = \begin{vmatrix} 231 \\ 312 \\ 123 \end{vmatrix}$$

Figure 1

$$\begin{vmatrix} (3,2) & (1,3) & (2,1) \\ (2,3) & (3,1) & (1,3) \\ (1,1) & (2,2) & (3,3) \end{vmatrix}$$

Figure 2

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mained the question of the existence of squares of order 6. Euler posed the problem as follows: Each of six different regiments has six officers, one belonging to each of six different ranks. Can these 36 officers be arranged in a square formation so that each row and file contains one officer of each rank, and one of each regiment? [Euler 1923, 292].

Euler showed that the problem of n^2 officers, which is identical to the problem of constructing a Graeco-Latin square of order n, can always be solved if n is odd or if n is an "evenly even" number (n is divisible by 4). After extensive trials, Euler stated: "I do not hesitate to conclude that it is impossible to produce any complete square of 36 cells, and the same possibility extends to the cases of $n = 10$, $n = 14$ and in general to all unevenly even numbers" [Euler 1923, 383]. This became the famous Euler conjecture. Stated in modern terminology the conjecture says that there does not exist a pair of orthogonal Latin squares of order $n = 4k + 2$ for any positive integer k.

The exhausting enumeration needed to verify Euler's conjecture (even for $n = 6$) is so great that it was not until 1900 that the French mathematician G. Tarry [1900] proved the conjecture for $n = 6$. As n increases the labor involved to verify Euler's conjecture becomes nearly overpowering, even with the use of computers.

Renewed interest in Latin squares was generated by Sir Ronald Fisher [1926], who studied these squares and applied them to the design of field experiments in agriculture. The principles of experimentation introduced by Fisher later found a wide field of application in such areas as biology, medicine, and industry. Fisher's work led him and F. Yates to enumerate all Latin squares of order 6 and to verify that there were no orthogonal pairs of this order [Fisher and Yates 1934].

After 1920, R. Bose and other mathematicians discovered many properties of Latin squares. H. F. MacNeish [1922], and later H. B. Mann [1942], showed that if $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where p_1, $p_2, \cdots, p_m$ are distinct primes and α_i are positive integers, then one can always construct a set of $n(k) = \min (p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_m^{\alpha_m}) - 1$ mutually orthogonal Latin squares of order k [MacNeish 1922, Mann 1942]. It was well known that there cannot exist more than $k - 1$ mutually orthogonal Latin squares of order k, and if $k = p^n$, where p is a prime, then $k - 1$ mutually orthogonal Latin squares can be constructed. Thus, if $N(k)$ denotes the maximum possible number of mutually orthogonal Latin squares of order k, $N(k) = k - 1$ if $k = p^n$. Hence, MacNeish proved that $N(k) \geq n(k)$.

The key to the verification of Euler's conjecture was MacNeish's [1922] conjecture that $N(k) = n(k)$, which he alleged to prove. The importance of MacNeish's conjecture is that, if true it would imply the correctness of Euler's conjecture about the nonexistence of two mutually orthogonal Latin squares of order $4k + 2$. This set the stage for the final chapter in this problem nearly 200 years old.

In 1938 F. W. Levi, Harding Professor and Chairman of the Department of Mathematics at the University of Calcutta, was discussing with Raj C. Bose some recent results established by the latter. Levi said that what Bose had proved in a recent paper [Bose 1938] had been partly established by MacNeish's paper of 1922, but Levi believed that part of MacNeish's proof was in error. It was. For, in 1958 E. T. Parker of Remington Rand Univac disproved MacNeish's conjecture by showing that there exist at least three mutually orthogonal squares of order 21 [Parker 1958] (according to MacNeish's conjecture there are at most two mutually orthogonal squares of order 21). While Parker's result did not disprove Euler's conjecture, it threw serious doubts on the validity of the conjecture.

Despite his work in many other areas of mathematics and statistics, Bose continued to be interested in Euler's conjecture. The impetus of Parker's work led Bose, who at this time was Professor of Statistics at the University of North Carolina, to develop some powerful general methods for the construction of large-order Graeco-Latin squares. In 1959, Bose and a student of his, S. S. Shrikhande, succeeded in applying these methods to the construction of a Graeco-Latin square of order 22 [Bose and Shrikhande 1959]. Since 22 is an even number not divisible by 4, Euler's conjecture was at long last proved false.

Bose and Shrikhande's method of constructing this counterexample to Euler's conjecture was based on the solution of a famous problem in recreational mathematics, called "Kirkman's School Girl Problem," proposed and solved by T. P. Kirkman [1850]. A school teacher wishes to take her fifteen girls for a walk once a day for a week. The girls walk three abreast in five rows. The problem is to arrange the girls so that for each of the seven days no girl will walk more than once in the same row with any other girl. The solution to this problem represents an example of an important type of experimental design called *balanced incomplete blocks.*

With the construction by Bose and Shrikhande of the counterexample to Euler's conjecture, the final disposition of the problem proceeded rapidly. Using other designs which had been investigated for the purpose of statistically controlled experiments, Bose and Shrikhande obtained an infinity of values of $n = 4k + 2$ for which Euler's conjecture is false. When Parker saw the results obtained by Bose and Shrikhande, he was able to develop a new method leading to his construction of an infinite series of Graeco-Latin squares of orders 10, 14, etc [Parker 1959]. Figure 3 shows the pair of orthogonal squares (i.e., a Graeco-Latin square) or order 10.

"At this stage," as Bose told me, "there ensued a correspondence between me and Shrikhande on the one hand, and Parker on the other. Methods were refined more and more, and it was ultimately established that Euler's conjecture is wrong for all values

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

Figure 3

$n = 4k + 2 > 6$. The suddenness with which complete success came in a problem which had baffled mathematicians for over one and three-quarters centuries startled the authors as much as anyone else. What makes this even more surprising is that the concepts employed were not even close to the frontiers of deep modern mathematics, and depended on finite groups, finite fields, difference sets, orthogonal arrays and imcomplete block designs."

There is a footnote to this story which I would like to add. First, the last sentence of Euler's memoir reads, "At this point I close my investigations on a question, which though of little use in itself, led us to rather important observations for the doctrine of combinations, as well as for the general theory of magic squares" [Euler 1923, 392]. This theory of magic squares has important practical applications, as we noted earlier in this paper. In fact, it was Fisher's application of Latin squares to problems in agriculture that initiated the process leading to the discovery that Euler's conjecture is false. In the words of Professor Bose, "It is a striking example of the unity of science, that the initial impulse which leads to a solution of the problem propounded by Euler, came from the practical needs of agricultural experimentation, thus reversing the more usual passage from basic to applied knowledge."

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